Ground-state fluctuations in finite Fermi systems

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We consider a small and fixed number of fermions in a trap. The ground state of the system is defined at $T=0$. For a given excitation energy, there are several ways of exciting the particles from this ground state. We formulate a method for calculating the number fluctuation in the ground state using microcanonical counting, and implement it for noninteracting fermions in harmonic confinement. This exact calculation for fluctuation, when compared with canonical or grand canonical ensemble averaging, gives considerably different results. This difference is expected to persist at low excitation even when the fermion number in the trap is large. For comparison, the well-known bosonic results are also given.

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I. INTRODUCTION

In this paper, we calculate the fluctuation in the population of confined fermions from the ground state of the system with excitation energy. This work was inspired by the recent experimental observation of quantum degeneracy in a dilute gas of trapped fermionic atoms at low temperatures by DeMarco and Jin [1,2]. Even before the experimental work, several theoretical papers had studied the properties of a trapped dilute gas of fermionic atoms. Butts and Rokhsar [3] studied the momentum and spatial distribution of the noninteracting system in the Thomas-Fermi approximation. Schneider and Wallis [4] looked into other thermodynamic properties of such a gas and the effect of shell structure on the specific heat. The effect of an attractive interaction on the low-temperature properties of a trapped fermi gas was investigated by Bruun and Burnett [5]. More recently, the collective excitations of the system in the normal phase have been examined by Bruun and Clark [6] and in the superfluid phase by Baranov and Petrov [7]. The latter paper also lists many other papers related to the superfluid phase of the trapped gas.

For simplicity, we consider noninteracting fermions in a confining potential. At zero temperature all the particles are in the ground state, occupying the lowest energy single-particle states up to the Fermi energy. A given excitation energy, however, may be shared in many different ways among the particles, so that the population of the original ground state is not fixed, although the total number N is still the same. Our objective here is to define and calculate this fluctuation in the ground-state occupation as a function of excitation energy or, equivalently, temperature. The corresponding problem for bosons in a trap has been studied by many groups [8] and will not be elaborated on here, although we shall also present the known bosonic results for comparison with fermions.

The traditional approach of determining the number fluctuations in a given quantum state relies on the statistical description of the system based on the grand canonical ensemble (GCE). As is well known, the GCE fails for bosons [9] as the temperature $T\to 0$, since the relative fluctuation tends to unity in this limit, rather than zero. For the fermionic problem under consideration, however—the limit $T\to 0$—the GCE ground-state fluctuation does go to zero, and the method yields results that are close to what one would obtain using the canonical ensemble. Our objective is to compare these results with an exact calculation. To this end, a combinatorial method based on microcanonical counting is developed in Sec. II, and calculations are made for particles in one- and two-dimensional harmonic traps. Comparison reveals substantial differences between the exact and the ensemble-averaged results. Unfortunately the exact fermionic calculations are very time consuming and were only performed for up to $N=15$ fermions. Nevertheless, as explained later, we expect this inaccuracy of the canonical ensemble averaging method to persist for fermions at low excitation, even when the fermion number in the trap is large. In Sec. III we show that even though the canonical entropies for noninteracting bosons and fermions in a one-dimensional harmonic trap are identical, the number fluctuations in the ground state are vastly different. The numerical results are discussed in Sec. IV.

II. FLUCTUATIONS IN THE INDEPENDENT PARTICLE MODEL

The GCE may be applied to obtain the ground-state fluctuation for fermions in a mean field with a set of single-particle orbitals. As is well known [10], the GCE fluctuation for the occupancy of a fermion in a given single-particle orbital i is given by

$$(\langle n_i^2\rangle-\langle n_i\rangle^2)^{1/2}=[\langle n_i\rangle(1-\langle n_i\rangle)]^{1/2}. \tag{1}$$

Here $\langle n_i\rangle$ is the usual Fermi occupancy factor at a given temperature T for the orbital i with energy ϵ_i:

$$\langle n_i\rangle=\frac{1}{\exp[(\epsilon_i-\mu)/T]+1}. \tag{2}$$

The chemical potential μ is determined by the condition that $\Sigma_i\langle n_i\rangle=N$, the total number of particles in the trap. To avoid

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complications, assume for the moment that there is no degeneracy. At $T=0$, the lowest orbitals ($i \leqslant F$) are occupied, where F is the Fermi level. In the GCE, consider this ground-state to be a subsystem that is in contact with the rest of the trap. At a nonzero temperature T, this subsystem gets depleted in number, and the ground-state fluctuation may be obtained by adding up the contributions of the orbitals ($i \leqslant F$) in Eq. (1). One may easily generalize this when there is degeneracy in the single-particle states, as in the two- or higher-dimensional harmonic oscillators.

We now proceed to develop the canonical ensemble method by assuming that the partition function is given analytically or may be computed. We first give the well-known recipe of calculating the N-particle quantum canonical partition system from this. This is done to define the canonical and the microcanonical multiplicities, and the exact ground-state fluctuation. These have already been done in connection to the bosonic problem [11,12] and apply equally well to fermions. We then go on to calculate the ground state fluctuation for fermions using combinatorics. The canonical partition function for bosons and fermions in any space dimension may be written as

$$Z_N^{B,F} = (\pm)^N \sum_{n_1, n_2, \ldots, n_N} \prod_{j=1}^{N} \frac{[\pm Z_1(j\beta)/j]^{n_j}}{n_j!}, \tag{3}$$

where $Z_1(\beta)$ is the single-particle partition function, β is the inverse temperature, and the upper and lower signs refer to bosons and fermions, respectively. The sum over the set of integers n_i is constrained by the relation

$$\sum_{j=1}^{N} j n_j = N. \tag{4}$$

The above formula allows us to write a general recursion relationship for the canonical partition function

$$Z_N^{B,F} = \frac{1}{N} \sum_{n=1}^{N} (\pm)^{n+1} Z_1(n\beta) Z_{N-n}^{B,F}(\beta) \tag{5}$$

for bosons $(+)$ and fermions $(-)$. We note that in the above recursion relation Z_0 is formally taken to be unity for consistency.

In order to perform explicit calculations, we specialize to the case of a harmonic oscillator in d dimensions. The single-particle partition function is given by

$$Z_1(\beta) = \left[\frac{x^{1/2}}{(1-x)}\right]^d, \tag{6}$$

where

$$x = \exp(-\beta\hbar\omega). \tag{7}$$

The canonical partition function for a system with N particles is then computed using Eq. (3) and is given by

$$Z_N = x^{Nd/2} P_N(x) \prod_{j=1}^{N} \frac{1}{(1-x^j)^d}, \tag{8}$$

where $P_N(x)$ is a polynomial in x which depends on the dimension and the statistics of the system. We shall use the notation $P_N(x) = B_N(x)$ and $P_N(x) = F_N(x)$ for bosons and fermions where necessary. The polynomial may be calculated using the recursion relation in Eq. (5):

$$P_N(x) = \frac{1}{N} \sum_{n=1}^{N} (\pm)^{n+1} \frac{\prod_{j=N-n+1}^{N} (1-x^j)^d}{(1-x^n)^d} P_{N-n}(x). \tag{9}$$

The recursion relation above should be used with the condition $P_0(x)=1$ for both bosons and fermions. We further note that in one dimension $B_N(x)=1$ for bosons and $F_N(x) = x^{N(N-1)/2}$ for fermions. They are, however, more complicated in higher dimensions [13].

A. Fluctuations from microcanonical counting

We first define the fluctuation in particle number from the ground state at a given excitation energy through a set of counting rules. Again we first write down the general formulas for a given a set of discrete energy levels and then specialize to the harmonic trap. The single-particle partition function may be written as

$$Z_1(\beta) = x^{\epsilon_0} \sum_{j=1}^{\infty} x^{\epsilon_j}, \tag{10}$$

where $x = e^{-\beta}$ and $\epsilon_j, j=0, \ldots, \infty$ are the single-particle energies. It is understood that β in the exponent defining x has been multiplied by a characteristic energy scale of the system, and similarly ϵ_j has been divided by the same, which we put to unity for convenience. For the harmonic oscillator, this energy scale is $\hbar\omega$, and x is given by Eq. (7). Substituting this into Eq. (3) and expressing Z_N in a power series in x, we obtain [11]

$$Z_N = x^{E_0} \sum_{k=1}^{\infty} \Omega(E_k^{(ex)}, N) x^{E_k^{(ex)}}, \tag{11}$$

where the N-particle eigenenergies $E_k = E_0 + E_k^{(ex)}$ form an ordered set, with E_0 and $E_k^{(ex)}$ denoting the ground-state energy and the excitation energy with respect to the ground state, respectively. The expansion coefficient $\Omega(E_k^{(ex)}, N)$ denotes the number of possible ways of distributing the excitation energy $E_k^{(ex)}$ in utmost N particles [12,11].

Furthermore, we may write $\Omega(E_k^{(ex)}, N)$ as

$$\Omega(E_k^{(ex)}, N) = \sum_{N_{ex}=1}^{N} \omega(E_k^{(ex)}, N_{ex}, N), \tag{12}$$

where $\omega(E_k^{(ex)}, N_{ex}, N)$ denotes the number of possible ways of distributing the excitation energy $E_k^{(ex)}$ among *exactly* N_{ex} particles. Hence the probability of exciting exactly N_{ex} particles from an N-particle system at an excitation energy $E_k^{(ex)}$ is given by

$$p(E_k^{(ex)},N_{ex},N)=\frac{\omega(E_k^{(ex)},N_{ex},N)}{\Omega(E_k^{(ex)},N)},\quad N_{ex}=0,\ldots,N. \tag{13}$$

By definition this probability is properly normalized. Further the probability has the following properties:

$$p(0,N_{ex},N)=\delta_{0N_{ex}}, \tag{14}$$

$$p(E_k^{(ex)},N_{ex},N)=0,\quad N_{ex}>N. \tag{15}$$

The number fluctuation in the ground state of the system may now be defined in terms of the moments of the probability distribution given above. We first define the moments

$$\langle N_{ex}\rangle=\sum_{N_{ex}=1}^{N} N_{ex}p(E_k^{(ex)},N_{ex},N), \tag{16}$$

$$\langle N_{ex}^2\rangle=\sum_{N_{ex}=1}^{N} N_{ex}^2p(E_k^{(ex)},N_{ex},N), \tag{17}$$

and the number fluctuation from the ground state is given by

$$\delta N_0^2=\langle N_{ex}^2\rangle-\langle N_{ex}\rangle^2 \\ =\langle N_0^2\rangle-\langle N_0\rangle^2, \tag{18}$$

since $\langle N_0\rangle+\langle N_{ex}\rangle=N$ is a constant.

A few remarks are in order here: The above definitions apply equally well to bosonic and fermionic systems. The fluctuation in the number of particles from the ground state is expressed here as a function of the excitation energy with respect to the ground state. In the case of bosons this is just the fluctuation from the lowest-energy single-particle state, where as for fermions it is the number fluctuation across the (zero-temperature) Fermi energy. Formally the above expressions complete the necessary basic definitions for further analysis. For a harmonic trap, we henceforth consider the excitation energy from the ground state to be $n\hbar\omega$, and denote the corresponding canonical and microcanonical multiplicities to be $\Omega(n,N)$ and $\omega(n,N_{ex},N)$. For bosons, it is well known [11] that the microcanonical multiplicity may be directly obtained by taking the difference between the canonical multiplicities of two different systems with N_{ex} and $N_{ex}-1$ particles:

$$\omega(n,N_{ex},N)=\Omega(n,N_{ex})-\Omega(n,N_{ex}-1). \tag{19}$$

This, however, is not true for fermions, as explained in the next section. Therefore we have to formulate a combinatorial method for counting the number of ways of exciting exactly N_{ex} fermions from the $T=0$ Fermi sea when n quanta of excitation energy are given to the system. This is a nontrivial problem. To the best of our knowledge, unlike the bose case, *no asymptotic formula is known for fermions*.

Consider a d-dimensional harmonic oscillator (closed shell). In a shell characterized by the index s, there are g_s single-particle orbitals i, each having the same energy $(s-1+d/2)\hbar\omega$. For simplicity, the fermions are taken to be spinless. In the ground state at $T=0$, the N fermions form a closed shell system filling an integral number of shells, $s=1,2,\ldots,S$, corresponding to the Fermi energy $E_F=(S-1+d/2)\hbar\omega$. The determination of the microcanonical distribution $\omega(n,N_{ex},N)$ depends not only on the distribution of N_{ex} fermions in the excited states (as in the case of bosons), but also on how the N_{ex} holes are distributed in the ground state. Let there be h_s holes $(h_s\leqslant g_s)$ in the shell s $(1\leqslant s\leqslant S)$, such that $\Sigma_{s=1}^{S}h_s=N_{ex}$. Then the number of ways these N_{ex} holes may be created in the ground state is given by

$$\prod_{s=1}^{S} {}^{(g_s)}C_{h_s}.$$

Now consider exciting N_{ex} particles from this ground state sharing n quanta of energy. An allowed configuration is one in which each and every one of these N_{ex} particles is found in states above the Fermi energy, with the shell indices ranging from $(S+1)$ up to $(S+n)$, such that their excitation energies add up to yield the total $E_{ex}=n\hbar\omega$. This complicates the counting rules for fermions as compared to bosons. We shall denote the occupancy of orbitals for the excited particles by m_i, where $i=S+1,\ldots,S+n$. The number of ways the m_i fermions are distributed in the state $S+i$ is then given by the counting rule

$${}^{(g_{S+i})}C_{m_i}.$$

The microcanonical distribution ω is then given by

$$\omega(n,N_{ex},N)=\sum_{\{m_i\}}\sum_{\{h_s\}}\prod_{s=1}^{S} {}^{(g_s)}C_{h_s}\prod_{i=S+1}^{S+n} {}^{(g_{S+i})}C_{m_i}, \tag{20}$$

where $N_{ex}=\Sigma_i m_i$, and the microcanonical multiplicity ω above is obtained by summing over all the allowed possibilities such that the sum total of the excited quanta is exactly n. Once the ω's are known the probability distribution may be calculated using Eq. (13) and hence the fluctuation as a function of the excitation energy.

We compare these results with the fluctuations obtained by the canonical ensemble averaging method of Parvan *et al.* [16] as detailed below. It is our objective to see how close the results are for the ground-state fluctuations calculated by the two methods.

B. Fluctuations from canonical ensemble averaging

In statistical thermodynamics, a macrostate at a given energy may be formed in many ways from the microstates, and the number of distinct ways is the multiplicity of the macrostate. As we saw in the preceding section (Sec. II A), the multiplicity $\Omega(E_k^{ex},N)$ was defined by Eq. (12) through this counting method. In the canonical ensemble, we may alternately define

$$\Omega(T,N)=\exp[S(T,N)]=\exp[(U-F)/T]=x^{-U}Z_N(x). \tag{21}$$

In the above, the internal energy $U(T)$ is determined as usual from the canonical partition function, and the excitation energy at any temperature is given by

$$E^{(ex)}=U(T)-U(0). \tag{22}$$

We may therefore compare the calculated quantities from the canonical and the microcanonical ensembles as a function of the excitation energy.

Comparing $\Omega(T,N)$ given by Eq. (21) from the canonical ensemble with the series (11), we see that it is as if only one term from this series is picked in the ensemble averaging. This is realized for a large number of particles, since the multiplicity $\Omega(E_k^{ex},N)$ increases rapidly with the excitation energy, where as the factor x^{E_k} decreases exponentially. In this paper, we focus on systems where N is not large, especially for fermions. It is therefore interesting to examine the differences in the results of the calculation made by the two methods. For bosons, one may obtain $\omega(T,N_{ex})$ directly from Eq. (19), where $\Omega(T,N_{ex})$ is determined by the canonical partition function of N_{ex} particles by replacing N by N_{ex} in Eq. (21). For fermions, however, the Fermi energy of a system defined by N_{ex} particles is less than the Fermi energy of the full system with N particles. The fluctuations are defined with respect to the ground state with the Fermi energy corresponding to the full system. Therefore Eq. (19) cannot be applied to fermions. A consistent definition of ground-state fluctuations applicable to both fermions and bosons, and which for bosons coincides with the earlier calculation with the definition (19), may, however, be given by the ensemble averaging method [15,16]. We summarize the method below.

The canonical partition function in Eq. (11) may be written in the occupation number representation as [16]

$$Z_N=\sum_{\{n_k\}}\prod_k x^{\epsilon_k n_k}, \tag{23}$$

where we have used the fact that the energy of the N-particle system for a given set of occupancies $\{n_k\}$ is given by $E=\Sigma_k\epsilon_k n_k$. The occupancy $n_k=0,1$ for fermions and may take any value up to N for bosons. At finite temperatures, using the recursion relation in Eq. (5) and some nontrivial algebra, the ensemble-averaged moments of the occupancy n_k may be written as [16]

$$\langle n_k\rangle_N=\frac{1}{Z_N}\sum_{j=1}^{N}(\pm)^{j+1}x^{j\epsilon_k}Z_{N-j}, \tag{24}$$

$$\langle n_k^2\rangle_N=\frac{1}{Z_N}\sum_{j=1}^{N}(\pm)^{j+1}jx^{j\epsilon_k}Z_{N-j}+\frac{1}{Z_N}\sum_{j=1}^{N}\sum_{i=1}^{N-j}(\pm)^{i+j}x^{(i+j)\epsilon_k}Z_{N-i-j}, \tag{25}$$

where the upper and lower signs refer to bosons and fermions, respectively. The subscript N in the occupancy and its second moment is to emphasize that these are in canonical ensemble averaging. The ground-state fluctuation is now given by

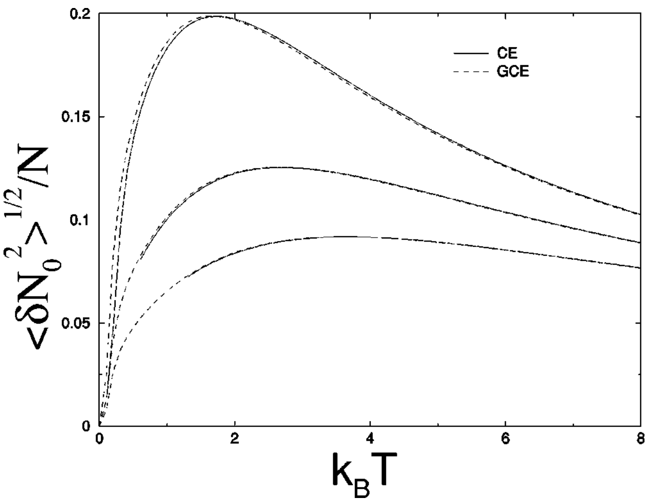

FIG. 1. Plots of $\langle(\delta N_0)^2\rangle^{1/2}/N$ versus temperature T (in units of $\hbar\omega$) as calculated by the GCE [Eq. (1)] and the canonical ensemble [Eq. (26)] for $N=6$, 15, and 28 fermions in a two-dimensional harmonic oscillator.

$$\delta N_0^2=\sum_k(\langle n_k^2\rangle_N-\langle n_k\rangle_N^2). \tag{26}$$

The sum runs through all the allowed k values in the ground state defined at zero temperature. For fermions, it can be shown [16] using Eqs. (24) and (25) that

$$\langle n_k^2\rangle_N=\langle n_k\rangle_N,$$

so that even in the canonical ensemble the fluctuation is given by the same form as Eq. (1) of the GCE. The difference lies in the fact that $\langle n_k\rangle_N$ is not given by the Fermi-Dirac (FD) distribution function (2). Numerical calculations show (see Fig. 1), however, that the canonical and the grand canonical methods give very similar results even when the particle number is very small. Both, however, differ from the exact microcanonical result. Before discussing the results, we describe below the especially interesting case of one-dimensional harmonic confinement. In this case, even though the canonical entropies for bosons and fermions are identical, the ground-state fluctuations for the two systems are very different. Moreover, the microcanonical ensemble yields substantially different results, especially for fermions.

III. FLUCTUATIONS IN A ONE-DIMENSIONAL HARMONIC TRAP

The one-dimensional harmonic trap is especially interesting because even though the canonical entropies for bosons and fermions are identical, the number fluctuations from the ground state are very different. This may be seen by writing the canonical N-particle partition function as

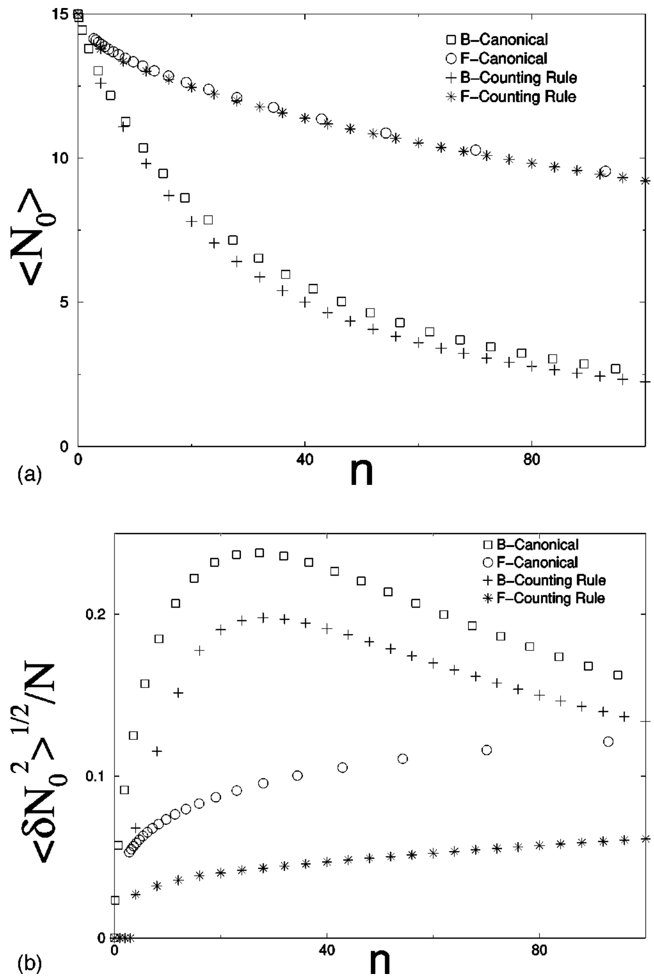

FIG. 2. (a) Plots of the ground-state occupancy $\langle N_0 \rangle$ versus the excitation energy n (in units of $\hbar\omega$) for $N=15$ bosons (fermions) in a one-dimensional harmonic oscillator. The results are displayed according to the legends in the inset. (b) Plots of the relative ground-state fluctuation $\langle \delta N_0^2 \rangle^{1/2}/N$ as a function of excitation quanta n for the same systems as in (a).

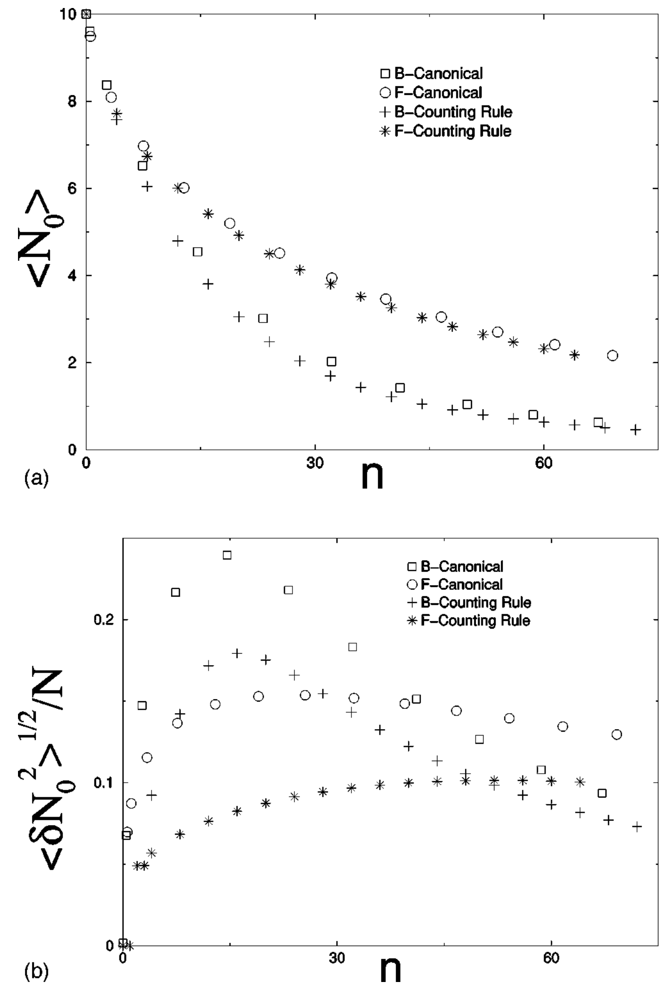

FIG. 3. (a) Plots of the ground-state occupancy $\langle N_0 \rangle$ versus the excitation quanta n for $N=10$ bosons (fermions) in a two-dimensional harmonic oscillator, according to the legends in the inset. (b) Plots of the relative ground-state fluctuation $\langle \delta N_0^2 \rangle^{1/2}/N$ as a function of excitation quanta n for the same systems as in (a).

$$Z_N = x^{gN(N-1)/2+N/2} \prod_{j=1}^{N} \frac{1}{(1-x^j)}, \tag{27}$$

with $g=0$ for bosons and $g=1$ for fermions. Actually, for other positive values of g, the above form is the exact partition function for the so-called Calogero-Sutherland model [14], where the N particles interact pairwise by a potential $(\hbar^2/m)g(g-1)\Sigma_{i<j}^{N}(x_i-x_j)^{-2}$. For bosons, the dimensionless parameter g is in the range $0 \leqslant g \leqslant 1/2$, while for fermions $g>1/2$. The special values $g=1(0)$ give noninteracting fermions (bosons). The effect of interaction has only been to shift the energy of every state by the same amount, which is absorbed in the prefactor. It follows from Eqs. (11) and (27) that we may write

$$Z_N = x^{gN(N-1)/2+N/2} \sum_{n=0}^{\infty} \Omega(n,N) x^n, \tag{28}$$

and that $\Omega(n,N)$ is independent of the parameter g, and is the same for bosons and fermions. Since it is the logarithm of $\Omega(n,N)$ that determines the canonical entropy of the system at an excitation energy of n quanta, it follows that the entropy is independent of g. The same result is true if one calculates the ensemble-averaged entropy at a given temperature. This may be easily verified by using the relation $F=-\ln Z_N/\beta$ for the free energy and then calculating the entropy $S=-\partial F/\partial T$.

For the microcanonical calculation of the fluctuation, we need to calculate the microcanonical multiplicity $\omega(n,N_{ex},N)$, which is the number of ways of distributing the n excitation quanta among exactly N_{ex} particles. Although the relation

$$\Omega(n,N) = \sum_{N_{ex}=1}^{N} \omega(n,N_{ex},N) \tag{29}$$

is obeyed both by fermions and bosons and the left-hand side of the above equation is the same for both, the microcanoni-

TABLE I. Tabulation of bosonic $\omega(n,N_{ex},N)$ for $N=3$.

	$n=$	1	2	3	4	5	6	7
	$N_{ex}=1$	1	1	1	1	1	1	1
		0	1	1	2	2	3	3
			(1+1)	(2+1)	(3+1)	(4+1)	(5+1)	(6+1)
					(2+2)	(3+2)	(4+2)	(5+2)
$\omega(n,N_{ex},N)$:	$N_{ex}=2$						(3+3)	(4+3)
		0	0	1	1	2	3	4
				(1+1+1)	(2+1+1)	(3+1+1)	(4+1+1)	(5+1+1)
						(2+2+1)	(3+2+1)	(4+2+1)
							(2+2+2)	(3+3+1)
								(3+2+2)
	$N_{ex}=3$							(3+2+2)
$\Omega(n,N)=$		1	2	3	4	5	7	8

cal counting of ω's is very different for the two cases. In the Appendix, this is illustrated explicitly for $N=3$ in Tables I and II. Thus the fluctuations for the bosonic and fermionic cases differ substantially when the exact counting method is used. As is to be expected, the fermionic fluctuation is considerably suppressed compared to the bosonic system. The same qualitative conclusion may also be reached by using Eqs. (24)–(26) based on canonical ensemble averaging rather than exact counting. In the next section, we display these results, as well as the results for the two-dimensional harmonic oscillator.

IV. RESULTS AND DISCUSSION

In Fig. 1, we compare the relative ground-state fermionic fluctuation, as obtained from the grand canonical [Eq. (1)] and the canonical [Eq. (26)] methods. The relative fluctuations for with $N=6$, 15, and 28 fermions in a two-dimensional harmonic oscillator are plotted as a function of temperature. We see that the agreement between the two methods is very good, especially as the particle number is increased. As was mentioned at the outset, this is in contrast to the bosonic case, where the GCE fails completely at very low temperatures due to the macroscopic occupancy of the ground state. For the one-dimensional harmonic oscillator, the agreement between the grand canonical and canonical results is even better, and is not shown separately.

In Fig. 2, we display (a) the ground-state occupancy $\langle N_0\rangle$ and (b) the relative fluctuation $\langle \delta N_0\rangle/N$ for $N=15$ noninteracting particles in a one-dimensional harmonic oscillator potential as a function of the excitation energy. The excitation energy, rather than the temperature, is chosen as the variable. This is because the exact microcanonical counting is done naturally as a function of the excitation energy, and relating it to temperature (unlike in canonical and grand canonical ensembles) would involve unnecessary approximations. Further, we do not show the GCE results, since these are almost identical with the canonical ones. Although the bosonic results are well known, these are also shown for comparison with the fermionic fluctuations. The canonical method gives results in close agreement with counting for the ground-state occupancy $\langle N_0\rangle$, but overestimates the relative fluctuation substantially. In one dimension, the number of microcanonical possibilities $\omega(n,N_{ex},N)$ is very restricted at low excitations due to the nondegeneracy in the single-particle energy levels and the Pauli exclusion principle. This results in (i) $\langle N_0\rangle$ for fermions getting depleted more slowly than bosons and (ii) reduced fluctuation. In Fig. 2, the same quantities are displayed for particles in a two-dimensional harmonic oscillator potential. It is well known from previous work [8,11,12] that there is a peak in the relative fluctuation somewhat below the critical temperature for bosons. For fermions, the graph has less structure, especially in the exact calculation. The peak in the bosonic fluctuation signals a phase transition, which is absent in noninteracting fermions.

The microcanonical method of exact counting for fermions is computationally very time-consuming because, unlike bosons, Eq. (19) cannot be used. We have therefore restricted the fermionic calculations to only up to $N=15$ particles. As we see from Figs. 2 and 3, there is considerable difference in the results for the relative fluctuations for small particle numbers. For fermions, we expect this difference to persist at low excitations even when N is large. This is because at low excitations, only a small fraction of the fermions near the Fermi sea can be excited, so the effective number of fermions contributing to the number fluctuation remains small even when the system is large. For bosons, there is no difficulty in performing the microcanonical calculation for a large number of particles.

We have made the comparison between the microcanonical and canonical results as functions of the excitation energy, rather than temperature. As stated earlier, the excitation energy is the natural variable for microcanonical counting. For the canonical ensemble, of course, a mapping from the excitation energy to temperature can be done using Eq. (22). We may also note that starting from the canonical equation (21) and using Eq. (19) for bosons, one can obtain the analytical expression for the low temperature dependence of the fluctuation, as, for example, given by Eq. (14) of [11]. It remains a challenging problem to obtain similar microcanonical relations for fermions.

TABLE II. Tabulation of fermionic $\omega(n,N_{ex},N)$ for $N=3$.

			n						
	L_1	n_{min}	1	2	3	4	5	6	7
$N_{ex}=1$	3_1	1	1	1	1	1	1	1	1
	2_1	2	0	1	1	1	1	1	1
	1_1	3	0	0	1	1	1	1	1
$\omega(n,1,N)=$			1	2	3	3	3	3	3
		4	0	0	0	1	1	2	2
						(1+3)	(1+4)	(1+5)	(1+6)
$N_{ex}=2$	$3_1 2_1$							(2+4)	(2+5)
		5	0	0	0	0	1	1	2
							(1+4)	(1+5)	(1+6)
	$3_1 1_1$								(2+5)
		6	0	0	0	0	0	1	1
	$2_1 1_1$							(2+4)	(2+5)
$\omega(n,2,N)=$			0	0	0	1	2	4	5
$N_{ex}=3$	$3_1 2_1 1_2$	9	0	0	0	0	0	0	0
$\omega(n,3,N)=$			0	0	0	0	0	0	0
$\Omega(n,N)=$			1	2	3	4	5	7	8

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V. APPENDIX

Calculation of $\omega(n,N_{ex},N)$ in one-dimensional harmonic oscillator.

A. Bosons

For illustration, in Table I, we display the simple case of $N=3$ bosons, with the number of excitation quanta $n\leqslant 7$. Since there is no degeneracy, $\omega(n,N_{ex},N)$ in this case is just the number of distinct ways in which the integer n may be partitioned amongst exactly N_{ex} identical bosons ($N_{ex}\leqslant 3$). In Table I, n increases from left to right, and increasing values of N_{ex} are tabulated in a vertical column. The integer in each box is the corresponding $\omega(n,N_{ex},N)$, with the distinct partitions of n listed in brackets below it. For example, in the box under $(n=4,N_{ex}=2)$, we see that $\omega(4,2,3)=2$, and the two distinct partitions of 4 are $(3+1)$ and $(2+2)$. In the last row is listed $\Omega(n,N)$, which is obtained by adding the $\omega(n,N_{ex},N)$'s in each vertical column. Note that these check with the coefficients in the expansion of the three-boson canonical partition function:

$$Z_3=1+x+2x^2+3x^3+4x^4+5x^5+7x^6+8x^7+\cdots .$$

B. Fermions

This case is the same as for bosons, except that two more vertical columns have been added under the headings L_1 and n_{min} (see Table II). For three spinless fermions, the lowest three energy levels (1,2,3) (in increasing order of energy) are occupied at $T=0$. The column under L_1 lists the possible configuration of holes in these levels for an excitation energy of $n\geqslant n_{min}$ quanta, the latter being listed in the adjacent column. For example, for $N_{ex}=2$, $(3_1 2_1)$ under the column L_1 denotes a two-hole configuration, with one hole in level 3 and another in level 2. The minimum energy required for this is $n_{min}=4$ in units of $\hbar\omega$.

[1] B. DeMarco and D. S. Jin, Science **285**, 1703 (1999).
[2] M. J. Holland, B. DeMarco, and D. S. Jin, Phys. Rev. A **61**, 053610 (2000).
[3] D. A. Butts and D. S. Rokhsar, Phys. Rev. A **55**, 4346 (1997).
[4] J. Schneider and H. Wallis, Phys. Rev. A **57**, 1253 (1998).
[5] G. M. Bruun and K. Burnett, Phys. Rev. A **58**, 2427 (1998).
[6] G. M. Bruun and C. W. Clark, Phys. Rev. Lett. **83**, 5415 (1999).
[7] M. A. Baranov and D. S. Petrov, Phys. Rev. A **62**, 041601(R) (2000).
[8] See, for example, M. Holthaus and E. Kalinowski, Ann. Phys. (N.Y.) **276**, 321 (1999), and references therein.
[9] I. Fujiwara, D. ter Haar, and H. Wergeland, J. Stat. Phys. **2**, 329 (1970).
[10] C. Kittel and H. Kroemer, *Thermal Physics*, 2nd ed. (Freeman,

New York, 1980), p. 222.
[11] S. Grossmann and M. Holthaus, Phys. Rev. Lett. **79**, 3557 (1997).
[12] S. Grossmann and M. Holthaus, Phys. Rev. E **54**, 3495 (1996).
[13] H.-J. Schmidt and J. Schnack, Physica A **265**, 584 (1999).
[14] F. Calogero, J. Math. Phys. **10**, 2191 (1969); **10**, 2197 (1969); B. Sutherland, *ibid.* **12**, 246 (1971); **12**, 251 (1971).
[15] K. C. Chase, A. Z. Mekjian and L. Zamick, Eur. Phys. J. B **8**, 281 (1999).
[16] A. S. Parvan, V. D. Toneev, and M. Ploszajczak, Nucl. Phys. A **676**, 409 (2000).